Homework of Week 11

The deadline is 9:00am of January 5 (Sunday), 2020.

1. Prove that the Markov property of a random process $\{X_n\}_{n=0}^{\infty}$ is equivalent to the following property: for any $n > k \geq 0$, any $I \subseteq \{0, ..., k-1\}$, and any $x_0, ..., x_k, x_n$, it holds that

$$\Pr(X_n = x_n | X_k = x_k, ...X_0 = x_0) = \Pr(X_n = x_n | X_k = x_k, X_i = x_i, i \in I).$$

2. Given a Markov chain $\{X_n\}_{n=0}^{\infty}$, prove that for any $n > 0$ and any states $x_0, ..., x_n$,

$$\Pr(X_n = x_n, X_{n-1} = x_{n-1}, ..., X_1 = x_1 | X_0 = x_0) = \prod_{i=1}^{n} \Pr(X_i = x_i | X_{i-1} = x_{i-1}).$$

3. Assume state i of a Markov chain has period d_i. Prove $p_{ii}^{(kd)} > 0$ for any large enough k.
4. Given a Markov chain, let i and j be two states. Define $f_{ij}^{(k)}$ to be the probability that starting with state i at time 0, the chain first reaches state j at time k, and $p_{ij}^{(k)}$ be the k-step probability of reaching j from i. Prove that $p_{ij}^{(n)} = \sum_{i=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}$, where $p_{jj}^{(0)} = 1$.
5. Given a finite-state Markov chain, prove that at least one state is recurrent.
6. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 ... s_i ... s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.