Homework of Week 2

Deadline: 9:00am, November 3 (Sunday), 2019

1. Prove the following extensions of the Chernoff bound. Let $X = \sum_{i=1}^{n} X_i$, where the X_i's are independent Poisson trials. Let $\mu = \mathbb{E}[X]$. Choose any μ_L and μ_H such that $\mu_L \leq \mu \leq \mu_H$. Then, for any $\delta > 0$, $\Pr(X \geq (1+\delta)\mu_H) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu_H}$.

   Similarly, for any $0 < \delta < 1$, $\Pr(X \leq (1-\delta)\mu_L) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu_L}$.

2. Let $X_1, ...X_n$ be independent Poisson trials such that $\Pr(X_i = 1) = p_i$ and let $a_1, ...a_n$ be real numbers in $[0, 1]$. Let $X = \sum_{i=1}^{n} a_i X_i$ and $\mu = \mathbb{E}[X]$. Then the following Chernoff bound holds: for any $\delta > 0$, $\Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$. Also prove a similar bound for the probability $\Pr(X \leq (1-\delta)\mu)$ for any $0 < \delta < 1$.

3. A function f is said to be convex if it holds that $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ for any x_1, x_2 and $0 \leq \lambda \leq 1$.

   - Let Z be a random variable that takes on a finite set of values in $[0, 1]$, and let $p = \mathbb{E}[Z]$. Define the Bernoulli random variable X by $\Pr(X = 1) = p$ and $\Pr(X = 0) = 1 - p$. Show that $\mathbb{E}[f(Z)] \leq \mathbb{E}[f(X)]$ for any convex function f. (Hint: induce on the number of values that Z takes on, and apply the convexity.)
   - Use the fact that $f(x) = e^{tx}$ is convex for any fixed $t \geq 0$ to obtain a Chernoff-like bound for Z.

4. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 s_i s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.