

# Probabilistic Method and **Random Graphs**

## Lecture 1. Elementary probability theory with applications <sup>1</sup>

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<sup>1</sup>The slides are mainly based on Chapters 1 and 2 of *Probability and Computing*.

# Important information

## Course homepage

<https://probabilitycomputing.github.io>

## Teaching assistant

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Office hours: TBD

## Homework

Submit in **PDF** to: fanfanda@ict.ac.cn

**Deadline**: 9:00am, Sunday

## Grading policy

Homework+Participation: 50%

Final exam (Open book?): 50%

Warning: **Enrolling in this course is at your own risk!**

# The brilliant history of probability theory

Gamblers As long as human history?

Cardano 1564, "Book on Games of Chance"

Informal law of large numbers, sum of 3 dice

Fermat&Pascal 1654, fair division of the stake in an interrupted game of chance

Huygens 1657, comprehensive treatment, expectation values

Bernoulli 1713, *Ars Conjectandi*, a sound mathematical footing

Law of large numbers

de Moivre 1718, *The Doctrine of Chances*, a sound mathematical footing

Gauss 18xx, application in astronomy, normal distribution

Laplace 1812, *Theorie analytique des probabilites*,  
fundamental results: MGF, MLS, hypothesis testing

...

Kolmogorov 1933, *Foundations of the Theory of Probability*,  
modern axiomatic foundations

...

Laplace(1745-1827)

Probability theory is nothing but  
a formulation of common sense



Advice from this book: Part of the research process in random processes is first to **understand what is going on at a high level** and then to use this understanding in order to develop formal mathematical proofs. ...To gain insight, you should perform experiments based on **writing code to simulate** the processes.

# Why probability in CS: two fundamental ways

## Algorithm design

- Randomized
- Probability-theory-based: statistical, derandomized ...
- Quantum computing

## Algorithm analysis

- Average complexity
- Smoothed complexity: Spielman and Teng
- Learning theory

*No probability, no viability!*

# Probability axioms and basic properties

A probability space (modeling a random process) has 3 elements

Sample space  $\Omega \neq \emptyset$  The set of possible outcomes

Event family  $\mathcal{F} \subseteq 2^\Omega$  The set of eligible events, a  $\sigma$ -algebra

Prob. function  $\Pr : \mathcal{F} \rightarrow \mathbb{R}$  The *likelihood* of the events

$\Pr$  satisfies 3 conditions:

- $\text{Range}(\Pr) \subseteq [0, 1]$
- $\Pr(\Omega) = 1$
- $\Pr(\bigcup_{i \geq 1} E_i) = \sum_{i \geq 1} \Pr(E_i)$  if the **countably many** events are **mutually disjoint**

Remarks

- We mainly consider the discrete case
- Events are sets, so Venn diagrams will be used for intuition

## Coin flip

- $\Omega = \{H, T\}$
- $\mathcal{F} = 2^\Omega$
- $\Pr(H) = p, \Pr(T) = 1 - p$

$p = 1/2$  if the coin is unbiased.

# Union bound

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$$

## Inclusion-exclusion principle

$$\Pr\left(\bigcup_{i \geq 1}^n E_i\right) = \sum_{l=1}^n (-1)^{l-1} \sum_{i_1 < i_2 < \dots < i_l} \Pr\left(\bigcap_{r=1}^l E_{i_r}\right)$$

## Union bound (Boole's Inequality)

$$\Pr\left(\bigcup_{i \geq 1} E_i\right) \leq \sum_{i \geq 1} \Pr(E_i)$$

## Bonferroni Inequalities

- $\Pr\left(\bigcup_{i \geq 1}^n E_i\right) \leq \sum_{l=1}^r (-1)^{l-1} \sum_{i_1 < i_2 < \dots < i_l} \Pr\left(\bigcap_{r=1}^l E_{i_r}\right)$  for odd  $r$
- $\Pr\left(\bigcup_{i \geq 1}^n E_i\right) \geq \sum_{l=1}^r (-1)^{l-1} \sum_{i_1 < i_2 < \dots < i_l} \Pr\left(\bigcap_{r=1}^l E_{i_r}\right)$  for even  $r$



## Definition: independent events

- $\Pr(E \cap F) = \Pr(E) \Pr(F)$
- Events  $E_1, E_2, \dots, E_k$  are mutually independent if for any  $I \subseteq [1, k]$ ,  $\Pr(\bigcap_{i \in I} E_i) = \prod_{i \in I} \Pr(E_i)$

# Independence and conditional probability

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## Definition: conditional probability

- $\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}$ , well-defined if  $\Pr(F) \neq 0$
- Conditioning changes/restricts the sample space
- Probability changes when more information is available

# Independence and conditional probability

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## Corollary

- $\Pr(E|F) = \Pr(E)$  if  $E$  and  $F$  are independent
- Independence means that the probability of one event is not affected by the information on the other
- Chain rule:  $\Pr(\bigcap_{i=1}^n A_i) = \prod_{i=1}^n \Pr(A_i | \bigcap_{j=1}^{i-1} A_j)$

## Law of total probability

If  $E_1, E_2, \dots, E_n$  are mutually disjoint and  $\bigcup_{i=1}^n E_i = \Omega$ , then

$$\Pr(B) = \sum_{i=1}^n \Pr(B \cap E_i) = \sum_{i=1}^n \Pr(B|E_i) \Pr(E_i).$$

## Example

Find the probability that the sum of  $n$  dice is divisible by 6.

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## Example

Find the probability that the sum of  $n$  dice is divisible by 6.

## Solution:

- $X_k$ : the result of the  $k$ -th roll for  $1 \leq k \leq n$
- $Y_k = \sum_{i=1}^k X_i$  for  $1 \leq k \leq n$
- $\Pr(Y_n \equiv 0 \pmod{6}) = \sum_{i=1}^6 \Pr((Y_n \equiv 0 \pmod{6}) \cap (X_n = i))$
- Claim:  $\Pr((Y_n \equiv 0 \pmod{6}) \cap (X_n = i))$   
$$= \Pr(Y_{n-1} \equiv 6 - i \pmod{6}) \Pr(X_n = i)$$

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## Bayes' Law

If  $E_1, E_2, \dots, E_n$  are mutually disjoint and  $\bigcup_{i=1}^n E_i = \Omega$ , then

$$\Pr(E_j|B) = \frac{\Pr(B|E_j) \Pr(E_j)}{\Pr(B)} = \frac{\Pr(B|E_j) \Pr(E_j)}{\sum_{i=1}^n \Pr(B|E_i) \Pr(E_i)}.$$

# It is time to solve a BIG Problem!

- Monty Hall problem
- First appeared at *Ask Marilyn* column of Parade, 9.9.1990
- See the demo
- Named after the celebrated TV host Monty Hall
- Confusing, so that formal proofs are not convincing (Paul Erdos & Andrew Vazsonyi)
- What's your answer?

Marilyn in 2017



Monty in 1970'



# Solution to Monty Hall problem

## Proof

- Reference for a formal proof: The Monty Hall Problem, by Afra Zomorodian, 1998
- An intuitive proof: keeping for one door but switching for two



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## God is fair: smart Miss Marilyn made silly mistakes

- January 22, 2012: How likely are you chosen over one year?
- May 5, 2013: How many 4-digit briefcase combinations contain a particular digit?
- June 22, 2014: How many work hours is necessary?  
6 together, but a 4-hour gap for each
- January 25, 2015: Which salary options do you prefer?  
Annual \$1000 or semi-annual \$300 raises

## Random variable

- A real-valued function on the sample space of a probability space,  $X : \Omega \rightarrow R$
- Random variables on this same probability space have both functional operations and probability operations

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## Probability of a random variable

- $X = a$  stands for the event  $\{s \in \Omega | X(s) = a\}$
- $\Pr(X = a) = \sum_{s \in \Omega: X(s)=a} \Pr(s)$

# Random variables and expectation

## Random variable

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## Independent random variables

- $\Pr((X = x) \cap (Y = y)) = \Pr(X = x) \Pr(Y = y)$
- Gengerally,  $\Pr(\bigcap_{i \in I} (X_i = x_i)) = \prod_{i \in I} \Pr(X_i = x_i)$  for any  $I$

# Expectation: a basic characteristic

## Definition

- $\mathbb{E}[X] = \sum_{i \in \text{Range}(X)} i * \Pr(X = i)$
- It's finite if  $\sum_{i \in \text{Range}(X)} |i| * \Pr(X = i)$  converges

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## Linearity of expectation

- $\mathbb{E}[\sum_{i=1}^n a_i X_i] = \sum_{i=1}^n a_i \mathbb{E}[X_i]$
- No independence is required
- The only condition is that each  $\mathbb{E}[X_i]$  is bounded
- The most important property of expectation!

# Expectation: a basic characteristic

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## Product Counterpart

$\mathbb{E}[X * Y] = \mathbb{E}[X]\mathbb{E}[Y]$  if they are **independent**.

## Bernoulli random variable

- $\Pr(X = 1) = p, \Pr(X = 0) = 1 - p$
- Modeling coin flip
- $\mathbb{E}[X] = p * 1 + (1 - p) * 0 = p$
- $X^k = X$



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## An example

How many triangles among 4 nodes when the links appear independently randomly?

## Binomial random variable

- The number of successes in  $n$  independent trials of the Bernoulli experiment with success probability  $p$
- For any  $0 \leq i \leq n$ ,  $\Pr(X = i) = C_n^i p^i (1 - p)^{n-i}$
- $X = \sum_{i=1}^n X_i$
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## Application

Router packets sampling

## The story of Farmer&Rabbit

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### Geometric random variable

- The number of independent trials until success, where each trial has success probability  $p$
- $\Pr(X = i) = (1 - p)^{i-1}p$  for  $i \geq 1$
- $\mathbb{E}[X] = \sum_{i \geq 1} i(1 - p)^{i-1}p = 1/p$

# Geometric distribution

The story of Farmer&Rabbit

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Suppose the daily probability that God throws a rabbit at the trunk is  $10^{-4}$ . How many years does the farmer has to wait?

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### Memoryless: particular to geometric distribution

For geometric random variable  $X$ , if  $n > 0$ ,  
 $\Pr(X = n + k | X > k) = \Pr(X = n)$

## Problem statement

The # of boxes of milk to buy before you have all coupon types?



# Application: coupon collector's problem

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- $X_i$ : the number of boxes bought while you have  $i - 1$  types of coupons until you get the  $i$ th type
- $X = \sum_{i=1}^n X_i$
- $X_i$ : geometric random variable with parameter  $p_i = 1 - \frac{i-1}{n}$
- $\mathbb{E}[X_i] = \frac{1}{p_i} = \frac{n}{n-i+1}$

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$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = n \sum_{i=1}^n \frac{1}{i} = n \ln n + \Theta(n).$$

**Thanks!**