Probabilistic Method and Random Graphs

Lecture 7. Random Graphs [1]

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[1] The slides are mainly based on Lecture 13 of Ryan O'Donnell's lecture notes of *Probability and Computing* and Chapter 5 of the textbook *Probability and Computing*.

Questions, comments, or suggestions?

A recap of Lecture 6

Hashing

- Hash table: accurate, time-efficient, space-inefficient
- Info. fingerprint: small error, time-inefficient, space-efficient
- Bloom filter: small error, time-efficient, more space-efficient

Random Grpahs

- Axiomatic definition
  - Uniform random graph $\mathcal{G}_n$
- Generated by stochastic processes
  - Playing super dice
  - Erdös-Rényi model $\mathcal{G}_{n,p}$ proposed by Gilbert
  - $\mathcal{G}_{n,\frac{1}{2}} \sim \mathcal{G}_n$, statistics, homogeneity $\cdots$
  - Erdös-Rényi model $\mathcal{G}_{n,m}$
  - $\mathcal{G}_{n,m} \sim \mathcal{G}_{n,p} | (m \text{ edges exist})$

Reflection on $\mathcal{G}_{n,p}$

Homogeneity in degree

Degree of each vertex is $\mathrm{Bin}(n-1, p)$.
Highly concentrated, as proven

Dense for constant p

$m = \Theta(n^2)$ whp.
Billions of vertices with zeta edges, too dense

Unfit for real-world networks

Heterogeneous in degree distribution.
Sort of sparse

Remark

$\mathcal{G}_{n,p}$-type randomness does appear in big graphs

Szemerédi Regularity Lemma

Tool in extremal graph theory by Endre Szemerédi in 1970's

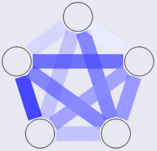
Hungarian-American (1940-)
Doctor vs Mathematician
Gelfond vs Gelfand

Szemerdi's Regularity Lemma

$\forall \epsilon, m > 0, \exists M > m$ such that any graph G with at least M vertices has an ϵ-regular k-partition, where $\exists m \leq k \leq M$.

Remark

Every large enough graph can be partitioned into a bounded number of parts which pairwise are like random graphs.



$$M = \left. m^{m^{m^{\cdot^{\cdot^{\cdot^{m}}}}}} \right\} d$$

$$\epsilon^{-\frac{1}{16}} \leq d = O(\epsilon^{-5})$$

A tentative model for sparse graphs

When the graph has constant average degree

Consider a social network with average degree 150 (Dunbar's #).
Let $p = \frac{150}{n}$. Does it work?

Too concentrated in degree

$D_i \sim \mathrm{Bin}(n-1, 150/n) \approx \mathrm{Poi}(150)$.
Union bound implies concentration around 150.
e.g. $\Pr(D_i \leq 25) \leq 25 \frac{e^{-150} 150^{25}}{25!} \approx 25 \times 10^{-36} < 10^{-34}$.

Random graphs with a given degree sequence

Degree sequence of an n-vertex graph G

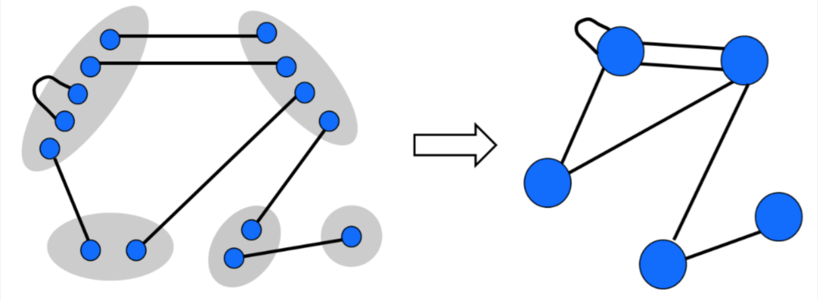
$n_0, n_1, ...n_n$ are integers.
$n_i =$ number of vertices in G with degree exactly i.
$\sum n_i = n, \sum i * n_i = 2m$

Random graphs with specified degree sequence

Introduced by Bela Bollobas around 1980.
Produced by a random process:
Step 1. Decide what degree each vertex will have.
Step 2. Blow each vertex up into a group of 'mini-vertices'.
Step 3. Uniformly randomly, perfectly match these vertices.
Step 4. Merge each group into one vertex.
Finally, fix multiple edges and self-loops if you like

Example

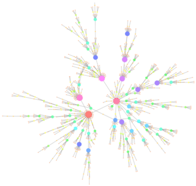
$n = 5, n_0 = 0, n_1 = 1, n_2 = 2, n_3 = 0, n_4 = 1, n_5 = 1$



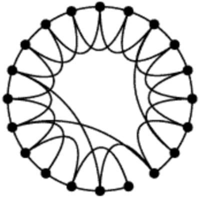
Other random graph models

Practical graphs are formed organically by "randomish" processes.

Preferential attachment model
Propsed by Barabasi&Albert in 1999
Scale-free network
First by Scottish statistician Udny Yule
in 1925 to study plant evolution



Rewired ring model
Propsed by Watts&Strogatz in 1998
Small world network



Threshold phenomena

Threshold: the most striking phenomenon of random graphs. Extensively studied in the Erdös-Rényi model $\mathcal{G}_{n,p}$.

Threshold functions

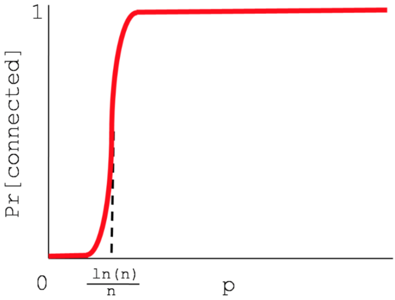
Given $f(n)$ and event E, if E does not happen on $\mathcal{G}_{n,o(f)}$ whp but happens on $\mathcal{G}_{n,w(f)}$ whp, $f(n)$ is a threshold function of E.

Sharp threshold functions

Given $f(n)$ and event E, if E does not happen on $\mathcal{G}_{n,cf}$ whp for any $c < 1$ but happens whp for any $c > 1$, $f(n)$ is a sharp threshold function of E.

Example

$f(n) = \frac{\ln n}{n}$ is a sharp threshold function for connectivity.



$f(n) = \frac{1}{n}$ is a sharp threshold function for giant component.

$f(n) = \frac{1}{n}$ is a threshold function for cycles.

Application: Hamiltonian cycles in random graphs

Objective

Find a Hamiltonian cycle if it exists in a given graph.
NP-complete, but ...
Efficiently solvable w.h.p. for $\mathcal{G}_{n,p}$, when p is big enough.

How?

A simple algorithm (use adjacency list model):

- Initialize the path to be a vertex.
- repeatedly use an unused edge to extend or rotate the path until a Hamiltonian cycle is obtained or a failure is reached.

Performance

Running time $\leq$ #edges $\Rightarrow$ inaccurate.
This does not matter if accurate w.h.p.
Challenge: hard to analyze, due to dependency.

A closer look at the algorithm

Essentially, extending or rotating is to sample a vertex.If an unseen vertex is sampled, add it to the path. When all vertices are seen, a Hamiltonian path is obtained, and almost end.

Familiar? Yes! Coupon collecting.
If we can modify the algorithm so that *sampling* at every step is uniformly random over all vertices, coupon collector problem results guarantee to find a Hamiltonian path in polynomial time. It is not so difficult to close the path.

Improvements

- Every step follows either unseen or seen edges, or reverse the path, with certain probability.
- Independent adjacency list (unused edges accessed by query), simplifying probabilistic analysis of random graphs

Modified Hamiltonian Cycle Algorithm

Under the independent adjacency list model

- Start with a randomly chosen vertex
- Repeat:
  - reverse the path with probability $\frac{1}{n}$
  - sample a used edge and rotate with probability $\frac{|used\_edges|}{n}$
  - select the first unused edge with the rest probability
- Until a Hamiltonian cycle is found or FAIL(no unused edges)

An important fact

Let V_t be the head of the path after the t-th step. If the unused_edges list of the head at time $t-1$ is non-empty, $\Pr(V_t = u_t | V_{t-1} = u_{t-1}, \ldots V_0 = u_0) = \frac{1}{n}$ for $\forall u_i$.

Coupon collector results apply: If no unused edges lists are exhausted, a Hamiltonian path is found in $O(n \ln n)$ iterations w.h.p., and likewise for closing the path.

Performance and Efficiency

Theorem

If in the independent adjacency list model, each edge (u, v) appear on u's list with probability $q \geq \frac{20 \ln n}{n}$, The algorithm finds a Hamiltonian cycle in $O(n \ln n)$ iterations with probability $1 - O(\frac{1}{n})$.

Basic idea of the proof

Fail $\Rightarrow$

- $\mathcal{E}_1$: no unused-edges list is exhausted in $3n \ln n$ steps but fail.
  - $\mathcal{E}_{1a}$: Fail to find a Hamiltonian path in $2n \ln n$ steps.
  - $\mathcal{E}_{1b}$: The Hamiltonian path does not get closed in $n \ln n$ steps.
- $\mathcal{E}_2$: an unused-edges list is exhausted in $3n \ln n$ steps.
  - $\mathcal{E}_{2a}$: $\geq 9 \ln n$ unused edges of a vertex are removed in $3n \ln n$ steps.
  - $\mathcal{E}_{2b}$: a vertex initially has $< 10 \ln n$ unused edges.

Proof: $\mathcal{E}_{1a}$ and $\mathcal{E}_{1b}$ have low probability

$\mathcal{E}_{1a}$: Fail to find a Hamiltonian path in $2n \ln n$ steps

The probability that a specific vertex is not reached in $2n \ln n$ steps is $(1 - 1/n)^{2n \ln n} \leq e^{-2 \ln n} = n^{-2}$.
By the union bound, $\Pr(\mathcal{E}_{1a}) \leq n^{-1}$.

$\mathcal{E}_{1b}$: The Hamiltonian path does not get closed in $n \ln n$ steps

$\Pr(\text{close the path at a specific step}) = n^{-1}$.
$\Rightarrow \Pr(\mathcal{E}_{1b}) = (1 - 1/n)^{n \ln n} \leq e^{-\ln n} = n^{-1}$.

Proof: $\mathcal{E}_{2a}$ and $\mathcal{E}_{2b}$ have low probability

$\mathcal{E}_{2a}$: $\geq 9\ln n$ unused edges of a vertex are removed in $3n\ln n$ steps

The number of edges removed from a vertex v's unused edges list $\leq$ the number X of times that v is the head.
$X \sim Bin(3n\ln n, n^{-1}) \Rightarrow \Pr(X \geq 9\ln n) \leq (e^2/27)^{3\ln n} \leq n^{-2}$.
By the union bound, $\Pr(\mathcal{E}_{2a}) \leq n^{-1}$.

$\mathcal{E}_{2b}$: a vertex initially has $< 10\ln n$ unused edges

Let Y be the number of initial unused edges of a specific vertex.
$\mathbb{E}[Y] \geq (n-1)q \geq 20(n-1)\ln n/n \geq 19\ln n$ asymptotically.
Chernoff bounds $\Rightarrow \Pr(Y \leq 10\ln n) \leq e^{-19(9/19)^2\ln n/2} \leq n^{-2}$.
Union bound $\Rightarrow \Pr(\mathcal{E}_{2b}) \leq n^{-1}$.

Altogether

$\Pr(fail) \leq \Pr(\mathcal{E}_{1a}) + \Pr(\mathcal{E}_{1b}) + \Pr(\mathcal{E}_{2a}) + \Pr(\mathcal{E}_{2b}) \leq \frac{4}{n}$.

The algorithm on random graph $\mathcal{G}_{n,p}$

Corollary

The modified algorithm finds a Hamiltonian cycle on random graph $\mathcal{G}_{n,p}$ with probability $1 - O(\frac{1}{n})$ if $p \geq 40\frac{\ln n}{n}$.

Proof

Define $q \in [0, 1]$ be such that $p = 2q - q^2$.
We have two facts:

- The independent adjacency list model with parameter q is equivalent to $\mathcal{G}_{n,p}$.
- $q \geq \frac{p}{2} \geq 20\frac{\ln n}{n}$.