hw2

Fanda Fan

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1 Problem 1

The proof of lower tail is omitted in the slide. It should be noted that

$$P(X \leq (1-\delta)\mu_L) = P(e^{-\lambda X} \geq e^{-\lambda(1-\delta)\mu_L})$$

Then use Markov's inequality to prove the rest.

2 Problem 2

For any $\lambda > 0$,

$$\begin{aligned} P(X \geq (1+\delta)\mu) =& P(e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}) \leq \frac{E[e^{\lambda X}]}{e^{\lambda(1+\delta)\mu}} = \frac{E[e^{\lambda \sum_{i=1}^{n} a_i X_i}]}{e^{\lambda(1+\delta)\mu}} \\ =& \frac{\prod_{i=1}^{n} E[e^{\lambda a_i X_i}]}{e^{\lambda(1+\delta)\mu}} \end{aligned}$$

$$E[e^{\lambda a_i X_i}] = 1 + p_i(e^{\lambda a_i} - 1) \leq e^{p_i(e^{\lambda a_i}-1)}$$

We need to prove that

$$\forall \lambda > 0, a_i \in [0,1], e^{\lambda a_i} - 1 \leq a_i(e^{\lambda} - 1)$$

Then we have

$$E[e^{\lambda a_i X_i}] = 1 + p_i(e^{\lambda a_i} - 1) \leq e^{p_i(e^{\lambda a_i}-1)} \leq e^{p_i a_i (e^{\lambda}-1)}$$

The rest will be easy.

3 Problem 3

1.

$$\begin{aligned} E[f(Z)] =& \sum_i p_i f(z_i) = \sum_i p_i f(z_i * 1 + (1-z_i) * 0) \\ \leq& \sum_i p_i z_i f(1) + p_i(1-z_i) f(0) = E(Z) f(1) + (1 - E(Z)) f(0) \\ =& p f(1) + (1-p) f(0) = E(f(X)) \end{aligned}$$

2. For any $t > 0, \delta > 0$,

$$P(Z \geq (1+\delta)p) = P(e^{tZ} \geq e^{t(1+\delta)p}) \leq \frac{E[e^{tZ}]}{e^{t(1+\delta)p}} = \frac{E[f(Z)]}{e^{t(1+\delta)p}} \leq \frac{E[f(X)]}{e^{t(1+\delta)p}}$$

The rest will be easy.